Chapter 9

Examining Exponential and Logarithmic Functions

In This Chapter

- Investigating exponential functions and rules of exponents
- Introducing laws of logarithms and simplifications
- Solving exponential and logarithmic equations

Exponential growth and decay are natural phenomena. They happen all around us. And, being the thorough, worldly people they are, mathematicians have come up with ways of describing, formulating, and graphing these phenomena. You express the patterns observed when exponential growth and decay occur mathematically with exponential and logarithmic functions.

Computing Exponentially

REMEMBER

An exponential function is unique because its variable appears in the exponential position and its constant appears in the base position. You write an *exponent,* or power, as a superscript just after the *base.* In the expression 3^x, for example, the variable x is the exponent, and the constant 3 is the base. The general form for an exponential function is $f(x) = a \cdot b^x$, where

- The base b is any positive number.
- The coefficient a is any real number (where $a \neq 0$).
- The exponent x is a variable representing any real number.

When you enter a number into an exponential function, you evaluate it by using the *order of operations,* evaluating the function in the following order:

1. **Powers and roots**
2. **Multiplication and division**
3. **Addition and subtraction**

Evaluate $f(x) = 4(3)^x + 1$ for $x = 2$ and $x = -2$.

Letting $x = 2$, you replace the x with the number 2. So, $f(2) = 4(3)^2 + 1 = 4(9) + 1 = 36 + 1 = 37$.

When $x = -2$, $f(-2) = 4(3)^{-2} + 1 = 4\left(\frac{1}{3^2}\right) + 1 = 4\left(\frac{1}{9}\right) + 1 = \frac{4}{9} + 1 = \frac{13}{9}$.

Getting to the Base of Exponential Functions

The base of an exponential function can be any positive number. The bigger the number, the bigger the function value becomes as the variable increases in value. (Sort of like the more money you have, the more money you make.) The bases can get downright small, too. In fact, when the base is some number between 0 and 1, you don't have a function that grows; instead, you have a function that falls.

Classifying bases

The base of an exponential function tells you so much about the nature and character of the function, making it one of the first things you should look for when working with exponential functions. One main distinguishing characteristic of bases of exponential functions is whether they're larger or smaller than 1. After you make that designation, you look at how *much* larger or how *much* smaller. The exponents also affect the expressions that contain them in somewhat predictable ways, making them another place to garner information about the function.

Because the domain of an exponential function is all real numbers, and the base is always positive, the result of b^x is always a positive number.

Focusing on bases

Algebra actually offers three classifications for the base of an exponential function, due to the fact that the numbers used as bases appear to react in distinctive ways when raised to positive powers:

- When $b > 1$, the values of b^x grow larger as x gets bigger — for example, $2^2 = 4$, $2^5 = 32$, $2^7 = 128$, and so on.
- When $b = 1$, the values of b^x show no movement. Raising the number 1 to higher powers always results in the number 1: $1^2 = 1$, $1^5 = 1$, $1^7 = 1$, and so on. You see no exponential growth or decay.
- When $0 < b < 1$, the value of b^x grows smaller as x gets bigger. Look at what happens to a fractional base when you raise it to the second, fifth, and eighth degrees: $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$, $\left(\frac{1}{3}\right)^5 = \frac{1}{243}$, $\left(\frac{1}{3}\right)^8 = \frac{1}{6{,}561}$. The numbers get smaller and smaller as the powers get bigger.

Examining exponents

When an exponent is replaced with a particular type of real number, you get results that are somewhat predictable. The exponent makes the result take on different qualities, depending on whether the exponent is greater than 0, equal to 0, or smaller than 0:

- When the base $b > 1$ and the exponent $x > 0$, the values of b^x get bigger and bigger as x gets larger — for example, $4^3 = 64$ and $4^6 = 4{,}096$. You say that the values grow *exponentially.*
- When the base $b > 1$ and the exponent $x = 0$, the only value of b^x you get is 1. The rule is that $b^0 = 1$ for any number except $b = 0$. So, an exponent of 0 really flattens things out.
- When the base $b > 1$ and the exponent $x < 0$ — a negative number — the values of b^x get smaller and smaller as the exponents get further and further from 0. Take these expressions, for example: $6^{-1} = \frac{1}{6}$ and $6^{-4} = \frac{1}{6^4} = \frac{1}{1{,}296}$. These numbers can get very small very quickly.

Introducing the more frequently used bases: 10 and e

Exponential functions feature bases represented by numbers greater than 0. The two most frequently used bases are 10 and e, where $b = 10$ and $b = e$.

It isn't too hard to understand why mathematicians like to use base 10 — in fact, just hold all your fingers in front of your face! All the powers of 10 are made up of ones and zeros — for instance, $10^2 = 100$, $10^9 = 1{,}000{,}000{,}000$, and $10^{-5} = 0.00001$. How much more simple can it get? Our number system, the decimal system, is based on tens.

Like the value 10, base e occurs naturally. Members of the scientific world prefer base e because powers and multiples of e keep creeping up in models of natural occurrences. Including e's in computations also simplifies things for financial professionals, mathematicians, and engineers.

If you use a scientific calculator to get the value of e, you see only some of e. The numbers you see estimate only what e is; most calculators give you seven or eight decimal places such as these first nine decimal places: $e \approx 2.718281828$.

Exponential Equation Solutions

The process of solving exponential equations incorporates many of the same techniques you use in algebraic equations — adding to or subtracting from each side, multiplying or dividing each side by the same number, factoring, squaring both sides, and so on.

Solving exponential equations requires some additional techniques, however. One technique you use when solving exponential equations involves changing the original exponential equation into a new equation that has matching bases. Another technique involves putting the exponential equation into a more recognizable form — such as a linear or quadratic equation — and then using the appropriate methods.

Creating matching bases

If you see an equation written in the form $b^x = b^y$, where the same number represents the bases b, then it must be true that $x = y$. You read the rule as follows: "If b raised to the xth power is equal to b raised to the yth power, that implies that $x = y$."

Solve the equation $2^{3+x} = 2^{4x-9}$ for x.

You see that the bases (the twos) are the same, so the exponents must also be the same. You just solve the linear equation $3 + x = 4x - 9$ for the value of x: $12 = 3x$, or $x = 4$. You then put the 4 back into the original equation to check your answer: $2^{3+4} = 2^{4(4)-9}$, which simplifies to $2^7 = 2^7$, or $128 = 128$.

Many times, bases are related to one another by being powers of the same number.

Solve the equation $4^{x+3} = 8^{x-1}$ for x.

You need to write both the bases as powers of 2 and then apply the rules of exponents. The number 4 is equal to 2^2, and 8 is 2^3, so you can write the equation as: $\left(2^2\right)^{x+3} = \left(2^3\right)^{x-1}$.

Now, raising a power to a power gives you $2^{2x+6} = 2^{3x-3}$.

The bases are the same, so set the exponents equal to one another and solve for x: $2x + 6 = 3x - 3$, which solves to give you $x = 9$. Substituting the 9 for x in the original equation, you get

$$4^{9+3} = 8^{9-1}$$
$$4^{12} = 8^{8}$$
$$16{,}777{,}216 = 16{,}777{,}216$$

Quelling quadratic patterns

When exponential terms appear in equations with two or three terms, you may be able to treat the equations as you do quadratic equations (see Chapter 3) to solve them with familiar methods. Using the methods for solving quadratic equations is a big advantage because you can factor the exponential equations, or you can resort to the quadratic formula.

You can make use of just about any equation pattern that you see when solving the exponential functions. If you can simplify the exponential to the form of a quadratic or cubic and then factor, find perfect squares, find sums and difference of squares, and so on, you've made life easier by changing the equation into something recognizable and doable.

Factoring out a common factor

When you solve a quadratic equation by factoring out a greatest common factor (GCF), you use the rules of exponents to find the GCF and divide the terms.

Solve for x in $3^{2x} - 9 \cdot 3^x = 0$.

Factor 3^x from each term and get $3^x(3^x - 9) = 0$. Now use the *multiplication property of zero* (MPZ; see Chapter 1) by setting each of the separate factors equal to 0.

$3^x = 0$ has no solution; 3 raised to a power can't be equal to 0. But the second factor does not equal 0.

$$\begin{aligned} 3^x - 9 &= 0 \\ 3^x &= 9 \\ 3^x &= 3^2 \\ x &= 2 \end{aligned}$$

The factor is equal to 0 when $x = 2$; you find only one solution to the entire equation.

Factoring a quadratic-like trinomial

The trinomial $5^{2x} - 26 \cdot 5^x + 25 = 0$, resembles a quadratic trinomial that you can factor using unFOIL. This exponential equation has the same pattern as the quadratic equation $y^2 - 26y + 25 = 0$, which would look something like the exponential equation if you replace each 5^x with a y.

Solve for x in the equation $5^{2x} - 26 \cdot 5^x + 25 = 0$.

The quadratic $y^2 - 26y + 25 = 0$ factors into $(y - 1)(y - 25) = 0$. Using the same pattern on the exponential version, you get the factorization $(5^x - 1)(5^x - 25) = 0$. Setting each factor equal to 0, when $5^x - 1 = 0$, $5^x = 1$. This equation holds true when $x = 0$, making that one of the solutions. Now, when $5^x - 25 = 0$, you say that $5^x = 25$, or $5^x = 5^2$. In other words, $x = 2$. You find two solutions to this equation: $x = 0$ and $x = 2$.

Looking into Logarithmic Functions

A *logarithm* is actually the exponent of a number. Logarithmic (abbreviated *log*) functions are the inverses of exponential functions. Logarithms answer the question, "What power gave me that answer?" The log function associated with the exponential function $f(x) = 2^x$, for example, is $f^{-1}(x) = \log_2 x$. The superscript –1 after the function name *f* indicates that you're looking at the inverse of the function *f*. So, $\log_2 8$, for example, asks, "What power of 2 gave me 8?"

A logarithmic function has a *base* and an *argument*. The logarithmic function $f(x) = \log_b x$ has a base *b* and an argument *x*. The base must always be a positive number and not equal to 1. The argument must always be positive.

You can see how a function and its inverse work as exponential and log functions by evaluating the exponential function for a particular value and then seeing how you get that value back after applying the inverse function to the answer. For example, first let $x = 3$ in $f(x) = 2^x$; you get $f(3) = 2^3 = 8$. You put the answer, 8, into the inverse function $f^{-1}(x) = \log_2 x$, and you get $f^{-1}(8) = \log_2 8 = 3$. The answer comes from the definition of how logarithms work; the 2 raised to the power of 3 equals 8. You have the answer to the fundamental logarithmic question, "What power of 2 gave me 8?"

Presenting the properties of logarithms

Logarithmic functions share similar properties with their exponential counterparts. When necessary, the properties of logarithms allow you to manipulate log expressions so you can solve equations or simplify terms. As with exponential functions, the base *b* of a log function has to be positive. I show the properties of logarithms in Table 9-1.

Table 9-1 Properties of Logarithms

Property Name	*Property Rule*	*Example*
Equivalence	$y = \log_b x \leftrightarrow b^y = x$	$y = \log_9 3 \leftrightarrow 9^y = 3$
Log of a product	$\log_b xy = \log_b x + \log_b y$	$\log_2 8z = \log_2 8 + \log_2 z$
Log of a quotient	$\log_b \frac{x}{y} = \log_b x - \log_b y$	$\log_2 \frac{8}{5} = \log_2 8 - \log_2 5$
Log of a power	$\log_b x^n = n\log_b x$	$\log_3 8^{10} = 10\log_3 8$
Log of 1	$\log_b 1 = 0$	$\log_4 1 = 0$
Log of the base	$\log_b b = 1$	$\log_4 4 = 1$

Exponential terms that have a base e have special logarithms just for the e's (the ease?). Instead of writing the log base e as $\log_e x$, you insert a special symbol, ln, for the log. The symbol ln is called the *natural logarithm,* and it designates that the base is e. The equivalences for base e and the properties of natural logarithms are the same, but they look just a bit different. Table 9-2 shows them.

Table 9-2 Properties of Natural Logarithms

Property Name	*Property Rule*	*Example*
Equivalence	$y = \ln x \leftrightarrow e^y = x$	$6 = \ln x \leftrightarrow e^6 = x$
Natural log of a product	$\ln xy = \ln x + \ln y$	$\ln 4z = \ln 4 + \ln z$
Natural log of a quotient	$\ln \frac{x}{y} = \ln x - \ln y$	$\ln \frac{4}{z} = \ln 4 - \ln z$
Natural log of a power	$\ln x^n = n\ln x$	$\ln x^5 = 5\ln x$
Natural log of 1	$\ln 1 = 0$	$\ln 1 = 0$
Natural log of e	$\ln e = 1$	$\ln e = 1$

As you can see in Table 9-2, the natural logs are much easier to write — you have no subscripts. Professionals use natural logs extensively in mathematical, scientific, and engineering applications.

Doing more with logs than sawing

You can use the basic exponential/logarithmic equivalence $\log_b x = y \leftrightarrow b^y = x$ to simplify equations that involve logarithms. Applying the equivalence makes the equation much nicer. If you're asked to evaluate $\log_9 3$, for example (or if you have to change it into another form), you can write it as an equation, $\log_9 3 = x$, and use the equivalence: $9^x = 3$. Now you have it in a form that you can solve for *x*. (The *x* that you get is the answer or value of the original expression.)

Evaluate $\log_9 3$.

After writing $\log_9 3 = x$, and the equivalence $9^x = 3$, you solve by changing the 9 to a power of 3 and then finding *x* in the new, more familiar form:

$$\begin{aligned}\left(3^2\right)^x &= 3\\ 3^{2x} &= 3^1\\ 2x &= 1\\ x &= \frac{1}{2}\end{aligned}$$

The result tells you that $\log_9 3 = \frac{1}{2}$ — much simpler than the original log expression.

Evaluate $10(\log_3 27)$.

First, write $\log_3 27 = x$ and its equivalence, $3^x = 27$. The number 27 is 3^3, so you can say that $3^x = 3^3$. For that statement to be true, it must be that $x = 3$. Now, replacing $\log_3 27$ with 3 in the original problem, you get $10(\log_3 27) = 10(3) = 30$. Another way to approach evaluating $\log_3 27$ is to write it as $\log_3 3^3$. Using the law of logarithms involving powers (refer to Table 9-1), the expression becomes $3\log_3 3$. Again, using a law of logarithms from the same table, you can substitute 1 for $\log_3 3$, so $3\log_3 3 = 3(1) = 3$.

Using log laws to expand expressions

A big advantage of logs is their properties and the way that you can change powers, products, and quotients into simpler addition and subtraction. Put all the log properties together, and you can change a single complicated expression into several simpler terms.

Simplify $\log_3 \frac{x^3\sqrt{x^2+1}}{(x-2)^7}$ by using the properties of logarithms.

First, use the property for the log of a quotient and then use the property for the log of a product on the new first term.

$$\log_3 \frac{x^3\sqrt{x^2+1}}{(x-2)^7} = \log_3 x^3\sqrt{x^2+1} - \log_3 (x-2)^7$$
$$= \log_3 x^3 + \log_3 \sqrt{x^2+1} - \log_3 (x-2)^7$$

The last step is to use the log of a power on each term, changing the radical to a fractional exponent first:

$$\log_3 x^3 + \log_3 (x^2+1)^{\frac{1}{2}} - \log_3 (x-2)^7 =$$
$$3\log_3 x + \frac{1}{2}\log_3 (x^2+1) - 7\log_3 (x-2)$$

The three new terms you create are each much simpler than the whole expression.

Using compacting

Results of computations in science and mathematics can involve sums and differences of logarithms. When this happens, you usually prefer to have the answers written all in one term, which is where the properties of logarithms come in.

Simplify $4\ln(x+2) - 8\ln(x^2-7) - \frac{1}{2}\ln(x+1)$ by writing the three terms as a single logarithm.

First, apply the property involving the natural log (ln) of a power to all three terms. Then factor out –1 from the last two terms and write them in a bracket:

$$\ln(x+2)^4 - \ln(x^2-7)^8 - \ln(x+1)^{\frac{1}{2}} =$$
$$\ln(x+2)^4 - \left[\ln(x^2-7)^8 + \ln(x+1)^{\frac{1}{2}}\right]$$

Now use the property involving the ln of a product on the terms in the bracket, change the $\frac{1}{2}$ exponent to a radical, and use the property for the ln of a quotient to write everything as the ln of one big fraction:

$$\ln(x+2)^4 - \left[\ln\left(x^2-7\right)^8 + \ln(x+1)^{\frac{1}{2}}\right] =$$

$$\ln(x+2)^4 - \left[\ln\left(x^2-7\right)^8 (x+1)^{\frac{1}{2}}\right] =$$

$$\ln(x+2)^4 - \ln\left(x^2-7\right)^8 \sqrt{x+1} =$$

$$\ln\frac{(x+2)^4}{\left(x^2-7\right)^8 \sqrt{x+1}}$$

The expression is messy and complicated, but it sure is compact.

Solving Equations Containing Logs

Logarithmic equations can have one or more solutions, just like other types of algebraic equations. What makes solving log equations a bit different is that you get rid of the log part as quickly as possible, leaving you to solve either a polynomial or an exponential equation in its place. Polynomial and exponential equations are easier and more familiar, and you may already know how to solve them. The only caution I present before you begin solving logarithmic equations is that you need to check the answers you get from the new, revised forms. You may get answers to the polynomial or exponential equations, but they may not work in the logarithmic equation. Switching to another type of equation introduces the possibility of *extraneous roots* — answers that fit the new, revised equation that you choose but sometimes don't fit in with the original equation.

Seeing all logs created equal

One type of log equation features each term carrying a logarithm in it (and the logarithms have to have the same base). You can apply the following rule:

If $\log_b x = \log_b y$, then $x = y$.

Solve the equation $\log_4 x^2 = \log_4(x + 6)$.

Apply the rule so that you can write and solve the equation $x^2 = x + 6$. Setting the quadratic equation equal to 0, you get $x^2 - x - 6 = 0$ which factors into $(x - 3)(x + 2) = 0$. The solutions $x = 3$ and $x = -2$ are for the quadratic equation, and both work in the original logarithmic equation. You always must check, though, because the solutions from a related quadratic equation don't always work in the original.

The following equation shows you how you may get an extraneous solution. Note that, when there's no base showing, you assume that you have common logarithms that are base 10.

Solve $\log(x - 8) + \log(x) = \log(9)$.

First apply the property involving the log of a product to get just one log term on the left: $\log(x - 8)(x) = \log(9)$. Next, you use the property that allows you to drop the logs and get the equation $(x - 8)x = 9$. This is a quadratic equation that you can solve with factoring. Multiplying on the left, you get $x^2 - 8x$. Subtracting 9 from each side, the quadratic equation is $x^2 - 8x - 9 = 0$, which factors into $(x - 9)(x + 1) = 0$. The two solutions of the quadratic equation are $x = 9$ and $x = -1$.

Checking the answers, you find that the solution 9 works just fine, but the –1 doesn't work: $\log(-1 - 8) + \log(-1) = \log(9)$. You can stop right there. Both of the logs on the left have negative arguments. The argument in a logarithm has to be positive, so the –1 doesn't work in the log equation (even though it was just fine in the quadratic equation). You determine that –1 is an extraneous solution and throw it out.

Solving log equations by changing to exponentials

When a log equation has log terms and a term that doesn't have a logarithm in it, you need to use algebra techniques and log properties (see Table 9-1) to put the equation in the form $y = \log_b x$. After you create the right form, you can apply the equivalence to change it to a purely exponential equation.

Solve $\log_3(x + 8) - 2 = \log_3 x$.

First subtract $\log_3 x$ from each side and add 2 to each side to get $\log_3(x + 8) - \log_3 x = 2$. Now you apply the property involving the log of a quotient, rewrite the equation by using the equivalence, and solve for x:

$$\begin{aligned}\log_3 \frac{x+8}{x} &= 2\\ 3^2 &= \frac{x+8}{x}\\ 9x &= x+8\\ 8x &= 8\\ x &= 1\end{aligned}$$

The only solution is $x = 1$, which works in the original logarithmic equation.